

THE GREEN'S FUNCTION FOR THE RADIAL SCHRAMM-LOEWNER EVOLUTION

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ABSTRACT. We prove the existence of the Green's function for radial SLE_κ for $\kappa < 8$. Unlike the chordal case where an explicit formula for the Green's function is known for all values of $\kappa < 8$, we give an explicit formula only for $\kappa = 4$. For other values of κ , we give a formula in terms of an expectation with respect to SLE conditioned to go through a point.

1. INTRODUCTION

The Schramm-Loewner evolution is a one-parameter family of two-dimensional random growth processes that was introduced in 1999 by the late Oded Schramm [10]. Denoting the parameter by $\kappa > 0$, these random growth processes (which are abbreviated as SLE_κ) have provided a valuable mathematical tool for obtaining rigorous results about a variety of discrete lattice models from statistical mechanics. Assuming appropriate boundary conditions, these discrete models contain a random curve which converges to SLE_κ for some value of κ (where κ depends on the model being considered). For instance, the exploration path for critical site percolation on the triangular lattice converges to SLE with $\kappa = 6$, contour lines in the discrete Gaussian free field converge to SLE with $\kappa = 4$, a perimeter curve for the uniform spanning tree converges to SLE with $\kappa = 8$, and cluster interfaces in the spin Ising and FK-Ising models at criticality converge to SLE with parameters $\kappa = 3$ and $\kappa = 16/3$, respectively. Moreover, loop-erased random walk converges to SLE_2 and there is strong evidence to suggest that self-avoiding walk converges to $SLE_{8/3}$. For more details see [4, 6] and references therein.

In addition to studying SLE as the scaling limit of interfaces in discrete statistical mechanics models, it is of intrinsic interest to understand the path properties of SLE; that is, properties of SLE as a continuous random process in the complex plane. For example, one might ask about the Hausdorff dimension of the SLE trace or the dimension of various random subsets of the trace, probabilities for different events such as the intersection of the trace with either a random or deterministic subset of the complex plane, or the distributions of certain functionals of the trace.

This paper studies the Green's function for radial SLE_κ , $0 < \kappa < 8$, from 1 to 0 in the unit disk \mathbb{D} which is the (normalized) probability that a radial SLE trace passes near a given point $z \in \mathbb{D}$. The Green's function for chordal SLE_κ from 0 to ∞ in the upper half plane \mathbb{H} , denoted by $\overline{G}_{\mathbb{H}}(z; 0, \infty)$, is well understood. It can be defined up to a multiplicative constant by the limit

$$(1) \quad \lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbb{P} \{ \Upsilon_\infty(z) \leq \epsilon \} = c^* \overline{G}_{\mathbb{H}}(z; 0, \infty)$$

where $d = 1 + \kappa/8$ is the Hausdorff dimension of the SLE trace $\gamma(0, \infty)$ and c^* is a constant. Here $\Upsilon_\infty(z)$ denotes one-half times the conformal radius of (the connected component containing z of) $\mathbb{H} \setminus \gamma_\infty$ with respect to z and γ_∞ denotes the trace $\gamma[0, \infty)$. It follows from the Koebe one-quarter theorem and Schwarz lemma that

$$\frac{1}{2} \Upsilon_\infty(z) \leq \text{dist}(z, \gamma_\infty \cup \mathbb{R}) \leq 2 \Upsilon_\infty(z).$$

Furthermore, the exact values of c^* and \overline{G} are known, see Section 2.1. (Of course, (1) only defines $c^*, \overline{G}_{\mathbb{H}}$ up to a multiplicative constant, but there is a natural choice of the constant that makes $\overline{G}_{\mathbb{H}}$ simplest.) In this paper, we consider the analogue for radial SLE $_{\kappa}$.

The outline of the remainder of the paper is as follows. In Section 2.1 we review the basic facts about the Green's function for chordal SLE while in Section 2.2 we introduce the Green's function for radial SLE. In Section 3 we derive the partial differential equation for the Green's function for radial SLE in \mathbb{D} assuming that the Green's function exists. This PDE can be solved explicitly when $\kappa = 4$ and so we state in Theorem 3.2 a formula for the Green's function for radial SLE $_4$ from 1 to 0 in \mathbb{D} , namely

$$G_{\mathbb{D}}(z; 1) = \sqrt{\frac{1 - |z|^2}{|z| \cdot |1 - z|^2}}$$

for $z \in \mathbb{D}$. In order to study the radial Green's function for other values of κ we work in the half-infinite cylinder \mathbb{H}^* obtained by taking the upper half plane \mathbb{H} and identifying points w_1, w_2 such that $w_1 - w_2 = k\pi$ for some integer k . The details are given in Section 4, and we state the main theorem, Theorem 4.3, which gives a representation of the radial SLE $_{\kappa}$ Green's function in \mathbb{H}^* for all $\kappa < 8$ in terms of an expectation with respect to SLE conditioned to go through a point. Finally, in Section 5 we prove the theorem by showing that the corresponding limit in (1) exists in the radial case.

2. THE GREEN'S FUNCTION FOR SLE

2.1. Chordal SLE Green's function. Chordal SLE $_{\kappa}$ for $\kappa > 0$ is a probability measure on curves γ connecting distinct boundary points of a domain D . It is characterized by conformal invariance and the domain Markov property. In the upper half plane it can be obtained by considering the Loewner equation

$$(2) \quad \partial_t g_t(z) = \frac{a}{g_t(z) - B_t}, \quad g_0(z) = z,$$

where $a = 2/\kappa$ and B_t is a standard one-dimensional Brownian motion. This equation is valid for all $t < T_z$ and the curve γ can be described by saying that $\{z : T_z > t\}$ is the unbounded component of $\mathbb{H} \setminus \gamma_t$ where $\gamma_t = \gamma[0, t]$. It also satisfies $g_t(\gamma(t)) = B_t$. It is known that for $0 < \kappa \leq 4$ the curves are simple, while for $d \geq 8$, the curves are plane filling. In this paper, we restrict to $\kappa < 8$.

Suppose d is the fractal dimension of the curve γ . Then we would expect that there is a function $\overline{G}(z)$ such that

$$\mathbb{P}\{\text{dist}(z, \gamma_{\infty}) \leq \epsilon\} \sim \overline{G}(z) \epsilon^{2-d} \quad \text{as } \epsilon \downarrow 0.$$

Rohde and Schramm [9] first noticed that if such a \overline{G} existed then

$$\overline{M}_t(z) = |g'_t(z)|^{2-d} \overline{G}(g_t(z) - B_t)$$

would have to be a local martingale. Using the scaling rule $\overline{G}(rz) = r^{d-2} \overline{G}(z)$ along with Itô's formula, they showed that this implies (up to a multiplicative constant) that

$$\overline{G}(z) = [\text{Im}(z)]^{d-2} \sin^{\frac{8}{\kappa}-1}(\arg z)$$

where $d = 1 + \kappa/8$. Using this idea Beffara [2] (see also [8]) proved that the Hausdorff dimension of the paths is d . It is still not known whether or not $\mathbb{P}\{\text{dist}(z, \gamma_{\infty}) \leq \epsilon\} \sim c \overline{G}(z) \epsilon^{2-d}$ for some c , but a similar result is known which we now describe. Let $\Upsilon_D(z)$ denote one-half the conformal radius of z in D . For simply connected domains D , which is all we will need, this means $\Upsilon_{\mathbb{H}}(z) = \text{Im}(z)$ and Υ transforms under conformal transformations by $\Upsilon_{f(D)}(f(z)) = |f'(z)| \Upsilon_D(z)$. Using the Schwarz lemma and the Koebe one-quarter theorem, one can see that

$$\frac{1}{2} \text{dist}(z, \partial D) \leq \Upsilon_D(z) \leq 2 \text{dist}(z, \partial D).$$

If D is not connected, we write $\Upsilon_D(z)$ for the conformal radius of z in the connected component of D containing z .

The Green's function for chordal SLE_κ in a simply connected domain D connecting boundary points w_1, w_2 is defined by

$$\overline{G}_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2} S_D(z; w_1, w_2)^{\frac{8}{\kappa}-1}.$$

Here $S_D(z; w_1, w_2) = \sin[\arg F(z)]$ where $F : D \rightarrow \mathbb{H}$ is a conformal transformation with $F(w_1) = 0$, $F(w_2) = \infty$. Note that $\overline{G}(z) = \overline{G}_{\mathbb{H}}(z; 0, \infty)$. Moreover, if γ is an SLE curve from w_1 to w_2 in D ,

$$(3) \quad \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{P} \{ \Upsilon_{D \setminus \gamma_\infty}(z) \leq \epsilon \} = c^* \overline{G}_D(z; w_1, w_2),$$

where

$$(4) \quad c^* = 2 \left[\int_0^\pi \sin^{8/\kappa} x \, dx \right]^{-1}.$$

A proof of this can be found in Lemma 2.10 of [8]; we will review the proof in Section 5.

The chordal Green's function satisfies the scaling rule

$$\overline{G}_D(z; w_1, w_2) = |f'(z)|^{2-d} \overline{G}_{f(D)}(f(z); f(w_1), f(w_2)).$$

Moreover,

$$\overline{M}_t^D(z) := \overline{G}_{D \setminus \gamma_t}(z; \gamma(t), w_2)$$

is a local martingale.

2.2. Radial SLE Green's function. Recall that radial SLE_κ for $\kappa > 0$ is a probability measure on curves connecting a boundary point and an interior point. For ease, and without loss of generality, we will choose the interior point to be the origin. If D is a simply connected domain containing the origin and $w \in \partial D$, we define $G_D(z; w)$, the *Green's function for radial SLE_κ* for $\kappa < 8$, by

$$(5) \quad \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{P} \{ \Upsilon_{D \setminus \gamma_\infty}(z) \leq \epsilon \} = c^* G_D(z; w)$$

where γ is a radial SLE_κ path from w to 0 and c^* is the same constant as in (4). In Theorem 4.3 we establish the existence of a function satisfying (5). It suffices to prove existence for $D = \mathbb{D}$, $w = 1$. Indeed if (5) holds in this case, and $f : D \rightarrow \mathbb{D}$ is a conformal transformation with $f(0) = 0$, $f(w) = 1$, then conformal invariance shows that (3) holds with

$$G_D(z; w) = |f'(0)|^{2-d} G_{\mathbb{D}}(f(z); 1).$$

As in the chordal case, we note that if such a function exists, then

$$M_t^D(z) := G_{D \setminus \gamma_t}(z; \gamma(t))$$

is a local martingale. Indeed, we could have used this property as the definition (up to a multiplicative constant) of the Green's function. If G is sufficiently smooth, see Sections 3 and 4, Itô's formula gives a partial differential equation for G . In Section 5 we show that (5) holds. We summarize the results here.

We write \mathbb{H}^* for the half-infinite cylinder obtained by taking the upper half plane \mathbb{H} and identifying points w_1, w_2 such that $w_1 - w_2 = k\pi$ for some integer k . Let $\psi(z) = e^{2iz}$ which is a conformal transformation of \mathbb{H}^* onto $\mathbb{D} \setminus \{0\}$. Let $\varphi(z) = (z - i)/(z + i)$ which is a conformal transformation of \mathbb{H} onto \mathbb{D} . Let

$$(6) \quad f(z) = \psi^{-1} \circ \varphi(z) = \frac{1}{2i} \log \left[\frac{z - i}{z + i} \right]$$

and set $\mathbb{D}_r^+ = \{z \in \mathbb{H} : |z| < r\}$ so that f is a conformal transformation of \mathbb{D}_1^+ onto a domain $f(\mathbb{D}_1^+)$ with $f(0) = 0$, $f'(0) = 1$. We write $G(z) = G_{\mathbb{H}^*}(z; 0)$ for the radial SLE Green's function in \mathbb{H}^* which satisfies

$$(7) \quad G(z) = G(x + iy) = 2^{2-d} e^{-2(2-d)y} G_{\mathbb{D}}(e^{-2y+i2x}; 1).$$

In analogy with the chordal case, we will write

$$G_D(z; w) = \Upsilon_D(z)^{d-2} F_D(z; w),$$

where $F_D(z; w)$ is a conformal *invariant*. For chordal SLE_κ , invariance under $z \mapsto rz$ showed that $r^{2-d} \overline{G}(rz)$ is a function of $\arg z$, and hence the solution could be found from a one-variable differential equation. It is a little more complicated in the radial case. We will write

$$G(x + iy) = \Upsilon_{\mathbb{H}^*}(x + iy)^{d-2} F(z) = [\sinh y \cosh y]^{d-2} F(z).$$

The second equality can be obtained by conformal transformation, using $\Upsilon_{\mathbb{D}}(z) = (1 - |z|^2)/2$. There is no scaling relation that allows F to be written as a function of one real variable. At the moment, except in the case $\kappa = 4$, we do not have an explicit expression for G . However, we can write

$$(8) \quad G(z) = G(x + iy) = [\sinh y \cosh y]^{d-2} \Lambda(z)^{\frac{8}{\kappa}-1} \Phi(z),$$

where $\Lambda(z)$ is an analogue of $S(z) = \sin(\arg z)$, namely

$$(9) \quad \Lambda(z) = \Lambda(x + iy) = \frac{\sinh y \cosh y}{|\sin z|},$$

and $\Phi(z)$ is a correction term. We can write Φ as

$$(10) \quad \Phi(z) = \mathbb{E}^* [g'_T(0)^q], \text{ where } q = \frac{(4 - \kappa)(\kappa - 8)}{8\kappa}.$$

We quickly explain the notation above; see Section 4 for more detail. Suppose γ is a radial SLE_κ path from 1 to the origin. Let $\mathbb{E}^* = \mathbb{E}_w^*$ denote expectation with respect to the probability measure obtained by conditioning the path to go through the point $w = e^{2iz}$. See [5] for an explanation of how this is made precise using the Girsanov theorem to tilt (or weight) by a particular local martingale. Let T be the time at which this path reaches w , and then g_T is the conformal transformation of the component of $\mathbb{D} \setminus \gamma_T$ containing the origin to \mathbb{D} with $g_T(0) = 0$, $g'_T(0) > 0$.

Note that $\Phi \equiv 1$ if $\kappa = 4$. If $z = x + iy \in \mathbb{H}$ with $|x| \leq \pi/2$ and $|y| \leq 1$, then $G(z) \asymp \overline{G}(z)$, where on the left we write z for the corresponding point on the cylinder \mathbb{H}^* . Moreover, $G(z) \sim \overline{G}(z)$ as $z \rightarrow 0$.

3. THE PDE FOR THE GREEN'S FUNCTION FOR RADIAL SLE IN \mathbb{D}

For now and the rest of the paper we fix $\kappa < 8$ and write

$$a = \frac{2}{\kappa} > \frac{1}{4}.$$

Suppose that $\gamma : [0, \infty) \rightarrow \mathbb{D} \setminus \{0\}$ is a radial SLE_κ path with $\gamma(0) = 1$ and $\gamma(t) \rightarrow 0$ as $t \rightarrow \infty$. To be specific, let D_t denote the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin, and let $g_t : D_t \rightarrow \mathbb{D}$ be the conformal transformation with $g_t(0) = 0$ and $g'_t(0) = e^{2at}$. Then $g_t(z)$ satisfies the differential equation

$$\partial_t g_t(z) = 2a g_t(z) \frac{e^{i2B_t} + g_t(z)}{e^{i2B_t} - g_t(z)}, \quad g_0(z) = z,$$

where B_t is a standard one-dimensional Brownian motion. Note that $g_t(\gamma(t)) = e^{i2B_t}$ and $g'_t(0) = e^{2at}$. Moreover, if $\tilde{g}_t(z) = e^{-i2B_t} g_t(z)$, then

$$(11) \quad \partial_t \log g_t(z) = 4a\pi \mathcal{H}_{\mathbb{D}}(\tilde{g}_t(z)),$$

where $\mathcal{H}_{\mathbb{D}}$ is the Schwarz kernel for \mathbb{D} given by

$$(12) \quad \mathcal{H}_{\mathbb{D}}(z) = \mathcal{H}_{\mathbb{D}}(z, 1) = \frac{1}{2\pi} \frac{1+z}{1-z} = \frac{1}{2\pi} \frac{1-|z|^2}{|1-z|^2} + \frac{i}{2\pi} \frac{z-\bar{z}}{|1-z|^2}.$$

Note that $u_{\mathbb{D}}(z) = \text{Re}[\mathcal{H}_{\mathbb{D}}(z)]$ is the Poisson kernel for \mathbb{D} and $v_{\mathbb{D}}(z) = \text{Im}[\mathcal{H}_{\mathbb{D}}(z)]$ is its harmonic conjugate. Here, and in what follows, we write the derivative of the logarithm as a convenient shorthand, namely

$$(13) \quad \partial_t \log g_t(z) = \frac{\partial_t g_t(z)}{g_t(z)}.$$

Since locally we can take a logarithm of g_t , there is no difficulty taking a branch cut so that the left side of (13) is well-defined. Consequently, the stochastic differential equation for $\tilde{g}_t(z)$ is

$$(14) \quad d \log \tilde{g}_t(z) = 4a\pi \mathcal{H}_{\mathbb{D}}(\tilde{g}_t(z)) dt - 2i dB_t.$$

Moreover, taking spatial derivatives (using $'$ to denote ∂_z) of (11) implies that

$$(15) \quad \begin{aligned} \partial_t \log g'_t(z) &= 4a\pi [\mathcal{H}_{\mathbb{D}}(\tilde{g}_t(z)) + \tilde{g}_t(z) \mathcal{H}'_{\mathbb{D}}(\tilde{g}_t(z))], \\ \partial_t \log |\tilde{g}'_t(z)| &= 4a\pi [u_{\mathbb{D}}(\tilde{g}_t(z)) + \text{Re} [\tilde{g}_t(z) \mathcal{H}'_{\mathbb{D}}(\tilde{g}_t(z))]] , \end{aligned}$$

$$(16) \quad \frac{\partial_t |\tilde{g}'_t(z)|^{2-d}}{(4a-1)\pi |\tilde{g}'_t(z)|^{2-d}} = u_{\mathbb{D}}(\tilde{g}_t(z)) + \text{Re} [\tilde{g}_t(z) \mathcal{H}'_{\mathbb{D}}(\tilde{g}_t(z))].$$

We write $G_{\mathbb{D}}(z) = G_{\mathbb{D}}(z; 1)$. We will find the PDE for $G_{\mathbb{D}}$ such that $M_t^{\mathbb{D}} := |\tilde{g}'_t(z)|^{2-d} G_{\mathbb{D}}(\tilde{g}_t(z))$ is a local martingale. Suppose that we write $z = re^{i\theta} \in \mathbb{D}$ and

$$\log \tilde{g}_t(z) = U_t + iV_t.$$

As a consequence of (14), we find

$$\partial_t U_t = 4a\pi u_{\mathbb{D}}(\tilde{g}_t(z)), \quad dV_t = 4a\pi v_{\mathbb{D}}(\tilde{g}_t(z)) dt - 2 dB_t.$$

Define H by $H(r, \theta) = r^{2-d} G_{\mathbb{D}}(re^{i\theta})$. Assuming sufficient smoothness on H and using Itô's formula, we see that $M_t^{\mathbb{D}}$ is a local martingale if and only if H satisfies

$$(17) \quad H_{\theta\theta} + aFH_{\theta} + aJH_r + \left(a - \frac{1}{4}\right) F_{\theta}H = 0,$$

where

$$F(r, \theta) = \frac{2r \sin \theta}{1 + r^2 - 2r \cos \theta}, \quad J(r, \theta) = \frac{r(1 - r^2)}{1 + r^2 - 2r \cos \theta}.$$

One might hope to find a solution of (17) the form

$$(18) \quad H = H(r, \theta) = u_{\mathbb{D}}(r, \theta)^p = \left(\frac{1 - r^2}{1 + r^2 - 2r \cos \theta} \right)^p$$

for some p . If H is of form (18) then

$$(19) \quad H_{\theta\theta} + aFH_{\theta} + aJH_r + \left(a - \frac{1}{4}\right) F_{\theta}H = (a - p)FH_{\theta} + \left(ap - p + a - \frac{1}{4}\right) F_{\theta}H.$$

The right side of (19) equals 0 only if $ap - p + a - \frac{1}{4} = 0$ and $a - p = 0$. Since the only solution of this pair of equations is $p = a = 1/2$, we conclude that $H(r, \theta)$ given by (18) satisfies (17) if and only if $p = a = 1/2$. Since $a = 1/2$ corresponds to $\kappa = 4$ (in which case $2 - d = 1/2$) we have now found an explicit formula for the Green's function (up to multiplicative constant) for radial SLE₄ from 1 to 0 in \mathbb{D} :

$$G_{\mathbb{D}}(z) = \sqrt{\frac{1 - |z|^2}{|z| \cdot |1 - z|^2}} = r^{-1/2} \left(\frac{1 - r^2}{1 + r^2 - 2r \cos \theta} \right)^{1/2},$$

where $z = re^{i\theta}$.

It is certainly true that $H(r, \theta) = u_{\mathbb{D}}(r, \theta)^p$ is not the only natural guess for the solution of (17). One other guess might be $u_{\mathbb{D}}(r, \theta)^p v_{\mathbb{D}}((r, \theta))^q$ for some values of p, q . It turns out that this guess does not lead to anything useful for analysis. In fact, we will see in Section 4 that \mathbb{H}^* is the best coordinate system for analyzing the Green's function for radial SLE.

4. RADIAL SLE_{κ} IN THE CYLINDER \mathbb{H}^*

In the previous section, we saw that it was easier to write the equation in polar coordinates. We continue this here; we will write points in \mathbb{D} as e^{2iz} where $z \in \mathbb{H}^*$. Radial SLE_{κ} in \mathbb{H}^* can be given as the solution to the Loewner equation

$$(20) \quad \partial_t h_t(z) = a \cot(h_t(z) + B_t), \quad h_0(z) = z,$$

where B_t is a standard one-dimensional Brownian motion. If we now define $g_t(e^{2iz}) = \exp\{2ih_t(z)\}$, then the maps g_t are those of radial SLE_{κ} in \mathbb{D} with the parametrization chosen so that $g'_t(0) = e^{2at}$. We write $\tilde{h}_t(z) = h_t(z) + B_t$, so that

$$(21) \quad d\tilde{h}_t(z) = a \cot(\tilde{h}_t(z)) dt + dB_t.$$

By taking the spatial derivative of (21) we obtain

$$(22) \quad \tilde{h}'_t(z) = \exp \left\{ -a \int_0^t \csc^2 \tilde{h}_s(z) ds \right\},$$

and so

$$\partial_t |\tilde{h}'_t(z)|^{2-d} = \left(\frac{1}{4} - a \right) \rho(z) |\tilde{h}'_t(z)|^{2-d}$$

where $\rho(z) = \rho(x, y)$ is given by

$$(23) \quad \rho(x, y) = \text{Re}(\csc^2 z) = \frac{\sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y}{(\sin^2 x + \sinh^2 y)^2}.$$

If we fix z and write

$$Z_t = \tilde{h}_t(z) = X_t + iY_t,$$

then $X_t = X_t(z)$ and $Y_t = Y_t(z)$ satisfy

$$(24) \quad dX_t = a v_t dt + dB_t, \quad \partial_t Y_t = -a u_t,$$

where

$$u_t = u(X_t, Y_t) = \frac{\sinh Y_t \cosh Y_t}{|\sin Z_t|^2}, \quad v_t = v(X_t, Y_t) = \frac{\sin X_t \cos X_t}{|\sin Z_t|^2}.$$

Furthermore, let $u(z) = u_0$ and $v(z) = v_0$ so that

$$(25) \quad u(z) = u(x, y) = \frac{\sinh y \cosh y}{\sin^2 x + \sinh^2 y}, \quad v(z) = v(x, y) = \frac{\sin x \cos x}{\sin^2 x + \sinh^2 y}.$$

Note that $\rho(x, y) = -v_x(x, y)$. Recall from (7) that if $G_{\mathbb{D}}(z) = G_{\mathbb{D}}(z; 1)$ denotes the Green's function for radial SLE_{κ} in \mathbb{D} , then the Green's function for radial SLE_{κ} in \mathbb{H}^* is

$$G(z) = G(x + iy) = 2^{2-d} e^{-2(2-d)y} G_{\mathbb{D}}(e^{-2y+i2x}).$$

Itô's formula (at $t = 0$) tells us that if $G(z) = G(x, y)$ satisfies the partial differential equation

$$(26) \quad \frac{1}{2} G_{xx} + avG_x - auG_y + \left(\frac{1}{4} - a \right) \rho G = 0,$$

then $M_t = M_t(z)$ defined by

$$(27) \quad M_t = |\tilde{h}'_t(z)|^{2-d} G(\tilde{h}_t(z))$$

will be a local martingale. Even though (26) is just a coordinate transformation of (17), it is easier to analyze.

Lemma 4.1. *If $p = (4a - 1) - 2(2 - d)$, $\zeta = (4a - 1) - (2 - d)$, and the function $H = H(z) = H(x, y)$ is defined as*

$$H(z) = |\sin z|^p u(z)^\zeta$$

where u is given by (25), then H satisfies the differential equation

$$(28) \quad \frac{1}{2}H_{xx} + avH_x - auH_y + \left(\frac{1}{4} - a\right)\rho H + apH = 0$$

where v is given by (25) and $\rho = -v_x$ as in (23). In particular,

$$(29) \quad N_t = N_t(z) = e^{apt} |\tilde{h}'_t(z)|^{2-d} H(\tilde{h}_t(z))$$

is a local martingale satisfying $dN_t = (1 - 4a) v_t N_t dB_t$.

Proof. This is a straightforward, although tedious, computation using Itô's formula. \square

We will now write the local martingale N_t in a way which clearly indicates the analogy with the chordal case. Let $p = (4a - 1) - 2(2 - d)$ be as in the lemma and let q be as in (10). Set

$$\beta = ap = -2aq = \frac{(2a - 1)(4a - 1)}{2} = \frac{(4 - \kappa)(8 - \kappa)}{2\kappa^2},$$

and set

$$(30) \quad \tilde{\Upsilon}_t = \Upsilon_{\mathbb{H}^* \setminus \gamma_t}(z) = \frac{\sinh Y_t \cosh Y_t}{|\tilde{h}'_t(z)|},$$

$$(31) \quad \Lambda_t = \Lambda_t(z) = |\sin Z_t| u_t = \frac{\sinh Y_t \cosh Y_t}{|\sin Z_t|},$$

so that

$$(32) \quad N_t = N_t(z) = e^{apt} |\tilde{h}'_t(z)|^{2-d} H(\tilde{h}_t(z)) = e^{\beta t} \tilde{\Upsilon}_t^{d-2} \Lambda_t^{4a-1}$$

is a local martingale satisfying

$$dN_t = (1 - 4a) v_t N_t dB_t.$$

This is not a martingale because it “blows up” on the (measure zero) set of paths which reach z . This local martingale also appears in [3].

We now use the Girsanov theorem to consider a new measure \mathbb{P}^* on paths tilted by N_t . (See [5] for an explanation of what is meant by using the Girsanov theorem to tilt (or weight) by a local martingale.) The Girsanov theorem states that

$$dB_t = (1 - 4a) v_t dt + dW_t,$$

where W_t is a standard Brownian motion with respect to the new measure \mathbb{P}^* . In particular,

$$(33) \quad dX_t = (1 - 3a) v_t dt + dW_t, \quad \partial_t Y_t = -a u_t.$$

Finally, recalling that $g'_t(0) = e^{2at}$, let

$$(34) \quad \Phi(z) = \mathbb{E}^* [g'_T(0)^q] = \mathbb{E}^* [e^{2aqT}] = \mathbb{E}^* [e^{-\beta T}],$$

where

$$T = T(z) = \inf\{t : Z_t(z) = 0\},$$

and \mathbb{E}^* denotes expectation with respect to \mathbb{P}^* . The next lemma shows that Φ is well defined.

Lemma 4.2. *If $z \in \mathbb{H}^*$, then $\mathbb{P}^*\{T < \infty\} = 1$ and $\Phi(z) < \infty$. Moreover,*

$$\lim_{z \rightarrow 0} \Phi(z) = 1.$$

Proof. First we show the following estimate which is valid for all $\kappa < 8$: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|z| \leq \delta$, then $\mathbb{P}^*\{T > \epsilon\} \leq \epsilon$. To see this note that if $|Z_t|$ is small, then

$$v_t = \frac{X_t}{X_t^2 + Y_t^2} [1 + O(|Z_t|^2)], \quad u_t = \frac{Y_t}{X_t^2 + Y_t^2} [1 + O(|Z_t|^2)].$$

Hence the Loewner equation (33) near 0 looks like

$$(35) \quad dX_t = (1 - 3a) \frac{X_t}{X_t^2 + Y_t^2} dt + dB_t, \quad \partial_t Y_t = -a \frac{Y_t}{X_t^2 + Y_t^2}.$$

These are the equations for two-sided radial SLE $_{\kappa}$ (see, e.g., [8] for definition), and it is known that this reaches the origin in finite time. A scaling argument gives the corresponding result for two-sided radial, and we can compare to get our estimate. In the $\beta \geq 0$ case, i.e., $\kappa \leq 4$, this immediately shows that $\Phi(z) \rightarrow 1$ as $z \rightarrow 0$. We also trivially have $\Phi(z) < 1$ in this case, so the lemma is complete for $\beta > 0$. For the remainder we assume that $\xi = -\beta > 0$, i.e., $4 < \kappa < 8$.

We return to the equations (33) and let

$$\sigma = \inf\{t : X_t = 0\}.$$

We claim there exists c_1 such that $\mathbb{E}^*[e^{\xi\sigma}] \leq c_1$ for all z . To see this, we note that X_t reaches zero before R_t where R_t satisfies

$$dR_t = r \cot R_t dt + dB_t, \quad r = \max\{0, 1 - 3a\},$$

since the latter equation has a larger drift away from the origin. Hence it suffices to show that $\mathbb{E}^*[e^{\xi\tau}] < \infty$ where $R_0 = \pi/2$ and $\tau = \inf\{t : R_t = 0\}$. This is true provided that $\xi < \lambda$ where λ is the smallest eigenvalue of

$$\frac{1}{2} \varphi''(x) + r \cot(x) \varphi'(x) + \lambda \varphi(x) = 0.$$

This eigenvalue corresponds to the positive eigenfunction φ and one can verify easily that

$$\varphi(x) = [\sin x]^{1-2r}, \quad \lambda = \frac{1}{2} - r > \xi.$$

We now use the previous paragraph for the argument that $\Phi(z) \rightarrow 1$ as $z \rightarrow 0$. We use it to show that for every $r > 0$, there exists $\epsilon_r > 0$ such that if $|z| \leq \epsilon_r$ then $\mathbb{E}^*[e^{\xi T}] \leq 1 + r$. To see this, we consider excursions. Let $0 < \delta \leq 1/2$ with $e^{\xi\delta} + 2\delta \leq 1 + r$ and let $\tau_{-1} = 0, \rho_0 = 0$,

$$\tau_0 = T \wedge \inf\{t : t = \delta \text{ or } |Z_t| = 2\delta\},$$

and if $\tau_0 < T$,

$$\begin{aligned} \rho_{j+1} &= \inf\{t > \tau_j : X_t = 0\}, \\ \tau_{j+1} &= T \wedge \inf\{t > \rho_{j+1} : t = \rho_{j+1} + \delta \text{ or } |Z_t| = 2\delta\}. \end{aligned}$$

Then,

$$\mathbb{E}^*[e^{\xi T} \mid \tau_{j-1} \leq T < \tau_j] \leq e^{\xi\delta(j+1)} c_1^j.$$

We can find $\epsilon > 0$ such that if $|z| < \epsilon$, then

$$\mathbb{P}^*\{\tau_0 < T\} \leq \delta/(2c_1),$$

and hence, since Y_t decreases, for δ sufficiently small, if $j \geq 1$, then

$$\mathbb{E}^*[e^{\xi T}; \tau_{j-1} \leq T < \tau_j] = \mathbb{P}^*\{\tau_{j-1} \leq T < \tau_j\} \mathbb{E}^*[e^{\xi T} \mid \tau_{j-1} \leq T < \tau_j] \leq \delta^j.$$

Therefore, $\mathbb{E}^*[e^{\xi T}; T > \tau_0] \leq 2\delta$. Since $T \leq \delta$ on the event $\{T \leq \tau_0\}$, we conclude that

$$\mathbb{E}^*[e^{\xi T}] \leq e^{\xi\delta} + 2\delta \leq 1 + r,$$

verifying that $\Phi(z) \rightarrow 1$ as $z \rightarrow 0$.

Finally we use the last estimate to show that $\Phi(z) < \infty$ for all z . Combining the estimates of the last two paragraphs with the Markov property, we see that if $\text{Im}(z) \leq \epsilon_r$, then $\mathbb{E}^*[e^{\xi T}] \leq (1+r)c_1$. Using the deterministic estimate

$$-a \coth Y_t \leq \partial_t Y_t \leq -a \tanh Y_t,$$

we see that if $z = x + iy$, then

$$\cosh^{-1}[(\cosh y)e^{-at}] \leq Y_t \leq \sinh^{-1}[(\sinh y)e^{-at}].$$

In particular, there exists a deterministic function $\psi(y)$ such that $\inf\{t : Y_t = \epsilon_r\} \leq \psi(y)$ for $z = x + iy$, and hence

$$\Phi(x + iy) \leq e^{\beta \psi(y)} (1+r)c_1.$$

This completes the proof. \square

We are now able to state the formula for the Green's function for radial SLE_κ in \mathbb{H}^* as given by (8).

Theorem 4.3. *Suppose that $p = (4a - 1) - (2 - d)$, $\zeta = (4a - 1) - 2(2 - d)$, $\beta = ap$, and $H(z) = |\sin z|^p u(z)^\zeta$ where $u(z)$ is given by (25). The Green's function for radial SLE in \mathbb{H}^* is*

$$G(z) = H(z) \Phi(z),$$

where

$$\Phi(z) = \mathbb{E}^*[e^{-\beta T}]$$

as in (34). In other words, If $z \in \mathbb{H}^*$, then for $\tilde{\Upsilon}_\infty(z)$ defined by (30),

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{P}\{\tilde{\Upsilon}_\infty(z) \leq \epsilon\} = c^* G(z), \quad \text{where} \quad c^* = 2 \left[\int_0^\pi \sin^{8/\kappa} x \, dx \right]^{-1}.$$

If $\kappa = 4$, then $G(z) = H(z)$.

We end this section with a brief discussion of some of the ingredients that go into the proof. For small times, radial SLE in \mathbb{H}^* is very close to chordal SLE_κ in \mathbb{H} . Let us make a precise statement of this fact. For $0 < r \leq 1$, let

$$\lambda_r = \inf\{t : |\gamma(t)| = r\}.$$

The half disk of radius 1 about 0 in \mathbb{H}^* can be considered as the half disk in \mathbb{H} , and hence we can view radial SLE in \mathbb{H}^* up to time λ_r as living in \mathbb{H} . Let H_r be the unbounded component of $\mathbb{H} \setminus \gamma_{\lambda_r}$, and let $\Upsilon_{H_r}(z)$, $\tilde{\Upsilon}_{H_r}(z)$ denote (1/2 times) the conformal radius about z where H_r is considered a subdomain of \mathbb{H} and \mathbb{H}^* , respectively. If $|z| \leq r$,

$$(36) \quad \Upsilon_{H_r}(z) = \tilde{\Upsilon}_{H_r}(z) [1 + o_r(1)].$$

Here and for the remainder of this section, $o_r(1)$ will denote a term that goes to zero in r uniformly over all other variables.

Let us assume that γ is parametrized using the half plane capacity in \mathbb{H} ; that is, the maps g_t satisfy (2). Since the half disk of radius r has half plane capacity r^2 , we know that $\lambda_r \leq r^2/a$. Considered as a solution to a radial equation (20), the parameterization is slightly different, say $\tilde{\lambda}(t) = \lambda(\sigma(t))$. However,

$$\sigma(t) = t [1 + O(t)].$$

In particular, in the radial parametrization, $\lambda_r \leq (r^2/a) [1 + O(r)]$.

For $r \leq 1$, let \mathbb{P} and $\tilde{\mathbb{P}}$ denote the probability measures on $\gamma(s)$, $0 \leq s \leq \lambda_r$, given by chordal and radial SLE_κ respectively. We can think of these either as measures on curves modulo reparametrization or on parametrized curves; however, in the latter case, one must take the same parametrization (chordal or radial) in both cases. Then \mathbb{P} and $\tilde{\mathbb{P}}$ are mutually absolutely continuous with

$\tilde{\mathbb{P}} = \mathbb{P}[1 + o_r(1)]$; that is,

$$\left| \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} - 1 \right| = o_r(1).$$

The analysis of the chordal case shows that if $0 < \epsilon \leq 1/2$, then for $|z| \leq r/\log(1/r)$,

$$\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon \Upsilon_0(z)\} = \mathbb{P}\{\Upsilon_{H_r}(z) \leq \epsilon \Upsilon_0(z)\} [1 + o_r(1)].$$

Also, there exists $\alpha > 0$ such that

$$\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon \Upsilon_0(z)\} = c^* S(z)^{4a-1} \epsilon^{4a-1} [1 + O(\epsilon^\alpha)],$$

where in this case the $O(\cdot)$ is uniform over z and we recall $S(z) = \sin(\arg z)$. Therefore, there exists $\epsilon_r \downarrow 0$, such that for $\epsilon < \epsilon_r$,

$$\tilde{\mathbb{P}}\{\Upsilon_{H_r}(z) \leq \epsilon \Upsilon_0(z)\} = c^* S(z)^{4a-1} \epsilon^{4a-1} [1 + o_r(1)],$$

and hence by (36), if $|z| \leq r/\log(1/r)$, $\epsilon < \epsilon_r$, then

$$\tilde{\mathbb{P}}\{\tilde{\Upsilon}_{H_r}(z) \leq \epsilon \tilde{\Upsilon}_0(z)\} = c^* \Lambda(z)^{4a-1} \epsilon^{4a-1} [1 + o_r(1)],$$

where $\Lambda(z)$ is given by (9). It is not obvious at the moment, and we prove in Proposition 5.4, that if $|z| \leq r/\log(1/r)$, then

$$\tilde{\mathbb{P}}\{\tilde{\Upsilon}_{H_r}(z) \leq \epsilon \tilde{\Upsilon}_0(z)\} = \tilde{\mathbb{P}}\{\Upsilon_{\mathbb{H} \setminus \gamma_\infty}(z) \leq \epsilon \tilde{\Upsilon}_0(z)\} [1 + o_r(1)].$$

We will not give the details of the proof of the estimates in the last paragraph, but we will sketch it here. One can compare chordal SLE_κ from 0 to ∞ in \mathbb{H} and radial SLE from 0 to i in \mathbb{H} by tilting by a particular local martingale; see [6]. Radial SLE_κ in \mathbb{H}^* can be obtained from radial SLE_κ in \mathbb{H} from 0 to i by the (multiple valued) transformation f as in (6). In both cases, one sees that the driving function changes from a standard Brownian motion to one with a drift. Under our conditions, the drift is uniformly bounded, and since time is bounded by $O(r^{1/2})$, we can bound the Radon-Nikodym derivative in the change of measure.

5. PROOF OF THEOREM 4.3

We start by reviewing the proof of the existence of the Green's function in the chordal case because we will need to use some of these facts. Fix $z \in \mathbb{H}$ and let γ be a chordal SLE_κ path from 0 to ∞ in \mathbb{H} which has equations

$$(37) \quad dX_t = a \frac{X_t}{X_t^2 + Y_t^2} dt + dB_t, \quad dY_t = -a \frac{Y_t}{X_t^2 + Y_t^2} dt.$$

The corresponding local martingale is

$$M_t = M_t(z) = \Upsilon_t^{d-2} S_t^{4a-1},$$

where

$$\Upsilon_t = \Upsilon_{\mathbb{H} \setminus \gamma_t}(z), \quad S_t = S_{\mathbb{H} \setminus \gamma_t}(z; \gamma(t), \infty)$$

as in Section 2.1. Let $\tau_\epsilon = \inf\{t : \Upsilon_t = \epsilon \Upsilon_0\}$. We will now review the proof that

$$(38) \quad \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{P}\{\tau_\epsilon < \infty\} = c^* S(z)^{4a-1}.$$

We begin by observing that

$$\begin{aligned} \mathbb{P}\{\tau_\epsilon < \infty\} &= \mathbb{E}[1\{\tau_\epsilon < \infty\}] \\ &= (\epsilon \Upsilon_0)^{2-d} \mathbb{E}[M_{\tau_\epsilon} S_{\tau_\epsilon}^{1-4a}; \tau_\epsilon < \infty] \\ &= \epsilon^{2-d} S(z)^{4a-1} \overline{G}(z)^{-1} \mathbb{E}[M_{\tau_\epsilon} S_{\tau_\epsilon}^{1-4a}; \tau_\epsilon < \infty] \\ &= \epsilon^{2-d} S(z)^{4a-1} \mathbb{E}^*[S_{\tau_\epsilon}^{1-4a}]. \end{aligned}$$

Here we write \mathbb{E}^* for the measure obtained by tilting by the local martingale M_t . This shows that proving (38) is equivalent to showing that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^* [S_{\tau_\epsilon}^{1-4a}] = c^*.$$

This is proved using a one-dimensional stochastic differential equation. If the path is reparametrized so that $\Upsilon_{\sigma(t)} = e^{-2at} \Upsilon_0$ and $\Theta_t = \arg Z_{\sigma(t)}$, then Θ_t satisfies

$$(39) \quad d\Theta_t = (1 - 2a) \cot \Theta_t dt + dB_t,$$

where B_t denotes a standard one-dimensional Brownian motion. Note that $1 - 2a < 1/2$ and by comparison with a Bessel process we see that the process reaches 0 or π in finite time. This reflects the fact that $\Upsilon_\infty < \infty$; indeed, the time in the new parametrization at which the boundary is reached is $-(2a)^{-1} \log \Upsilon_\infty$. If $\hat{M}_t = M_{\sigma(t)}$, then

$$d\hat{M}_t = 4a [\cot \Theta_t] \hat{M}_t dB_t.$$

If we weight by the local martingale M_t (or, equivalently, by \hat{M}_t), then we get

$$(40) \quad d\Theta_t = 2a \cot \Theta_t dt + dW_t,$$

where W_t is a standard Brownian motion in the new measure \mathbb{P}^* . This weighted measure is sometimes called two-sided radial SLE $_\kappa$ from 0 to ∞ through z (stopped at $T = T_z$, the time it reaches z). Since $2a > 1/2$, we see that in the new measure the process survives forever; in other words, the weighted process hits z . The invariant probability density for the process (40) is

$$f(\theta) = \frac{c^*}{2} \sin^{4a} \theta, \quad 0 \leq \theta < \pi, \quad \text{where} \quad c^* = 2 \left[\int_0^\pi \sin^{4a} x dx \right]^{-1}.$$

Moreover, there exists $\lambda > 0$ such that if $p_t^*(\theta_1, \theta_2)$ denotes the transition density for the process, then for every t with $e^{-2at} \leq 1/2$,

$$p_t^*(\theta_1, \theta_2) = f(\theta_2) \left[1 + O(e^{-\lambda t}) \right],$$

where the error term $O(e^{-\lambda t})$ is uniform over θ_1, θ_2 .

The proof in \mathbb{H}^* is similar but with some added complications. Recall from (32) that the local martingale is

$$N_t = N_t(z) = e^{\beta t} \tilde{\Upsilon}_t^{d-2} \Lambda_t^{4a-1}$$

where $\tilde{\Upsilon}_t = \Upsilon_{\mathbb{H}^* \setminus \gamma_t}(z)$ and $\Lambda_t = \Lambda_t(z)$ as in (30). We now fix $z \in \mathbb{H}^*$ and allow constants in what follows to depend on z . For each $\epsilon > 0$, let

$$\tau_\epsilon = \inf \{t : \tilde{\Upsilon}_t = \epsilon \tilde{\Upsilon}_0\}.$$

What we need to show is that

$$\mathbb{P} \{ \tau_\epsilon < \infty \} = c^* \Phi(z) \Lambda(z)^{4a-1} \epsilon^{2-d} [1 + o(1)],$$

where $o(1)$ represents a term which may depend on z that goes to zero as $\epsilon \rightarrow 0$. In analogy with the chordal case, observe that

$$\begin{aligned} \mathbb{P} \{ \tau_\epsilon < \infty \} &= \mathbb{E} [1 \{ \tau_\epsilon < \infty \}] \\ &= \epsilon^{2-d} \tilde{\Upsilon}_0^{2-d} \mathbb{E} \left[N_{\tau_\epsilon} e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a}; \tau_\epsilon < \infty \right] \\ &= \epsilon^{2-d} \tilde{\Upsilon}_0^{2-d} N_0 \mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a} \right] \\ &= \Lambda(z)^{4a-1} \epsilon^{2-d} \mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a} \right], \end{aligned}$$

where \mathbb{E}^* denotes the measure obtained by tilting by the local martingale N_t . To prove the theorem, we need to show that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a} \right] = c^* \mathbb{E}^* \left[e^{-\beta T} \right] = c^* \Phi(z).$$

The basic reason this is true can be explained as follows. As $\epsilon \rightarrow 0$, the quantity Λ_{τ_ϵ} is very much like S_{τ_ϵ} and asymptotically it behaves as in the chordal case. The quantity fluctuates as $\epsilon \rightarrow 0$, but with a stationary distribution giving

$$\mathbb{E} \left[\Lambda_{\tau_\epsilon}^{1-4a} \right] \rightarrow c^*.$$

The quantity $e^{-\beta \tau_\epsilon}$ is a “macroscopic” quantity that does not change much when ϵ is small. Since Λ_{τ_ϵ} is fluctuating on the small scale, we can see that the two random variables are asymptotically independent,

$$\mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a} \right] \sim \mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \right] \mathbb{E}^* \left[\Lambda_{\tau_\epsilon}^{1-4a} \right] \rightarrow c^* \mathbb{E}^* [e^{-\beta T}].$$

To prove this, we first note that if V is an event, then

$$(41) \quad \mathbb{P}[\{\tau_\epsilon < \infty\} \cap V] = \Lambda(z)^{4a-1} \epsilon^{2-d} \mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a} 1_V \right].$$

Let

$$\rho = \rho_\epsilon = \inf \left\{ t : \tilde{\Upsilon}_t = \epsilon^{1/2} \right\}.$$

For $t \geq \rho$, let $\hat{\gamma}(t) = \hat{\gamma}^\epsilon(t) = \tilde{h}_\rho(\gamma(t))$,

$$\lambda = \lambda_\epsilon = \inf \left\{ t \geq \rho : |\hat{\gamma}(t)| = \epsilon^{1/8} \right\},$$

and let V_ϵ be the event

$$V_\epsilon = \left\{ |\tilde{h}_\rho(z)| \leq \epsilon^{1/3}, \lambda > \tau_\epsilon \right\}.$$

To prove the theorem, it suffices to show that

$$(42) \quad \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{P}[\{\tau_\epsilon < \infty\} \setminus V_\epsilon] = 0,$$

and

$$(43) \quad \mathbb{P}[\{\tau_\epsilon < \infty\} \cap V_\epsilon] \sim c^* \Phi(z) \Lambda(z)^{4a-1} \epsilon^{2-d}.$$

To prove (42), let

$$\sigma = \sigma_\epsilon = \inf \{ t : |\gamma(t) - z| \leq \epsilon^{1/2}/10 \}.$$

The Koebe one-quarter estimate implies that $c_1 \epsilon^{1/2} \leq \Upsilon_\sigma \leq \epsilon^{1/2}$. By considering the image of the line segment from z to $\gamma(\sigma)$ under the map \tilde{h}_σ and using the Beurling estimate, we see that there exists c_2 such that $|\tilde{h}_\sigma(z)| \leq c_2 \epsilon^{1/4}$. Since

$$|\tilde{h}_\rho(z)| = \frac{\text{Im}[\tilde{h}_\rho(z)]}{S_\rho(z)} \leq \frac{\text{Im}[\tilde{h}_\sigma(z)]}{S_\rho(z)} = \frac{S_\sigma(z) |\tilde{h}_\sigma(z)|}{S_\rho(z)},$$

then $|\tilde{h}_\rho(z)| \geq \epsilon^{1/3}$ implies that

$$S_\sigma(z) \geq c_2^{-1} S_\rho(z) \epsilon^{1/12}.$$

Using Proposition 5.3 below, we see that there exists $c_3 < \infty$ such that

$$\begin{aligned} \mathbb{P} \left\{ \tau_\epsilon < \infty, S_\sigma(z) \geq c_2^{-1} S_\rho(z) \log(1/\epsilon) \mid \gamma_\sigma \right\} \\ \leq c_3 \epsilon^{(1-4a)/12} \mathbb{P} \left\{ \tau_\epsilon < \infty \mid \gamma_\sigma \right\}. \end{aligned}$$

Therefore,

$$\mathbb{P} \left\{ \tau_\epsilon < \infty, |\tilde{h}_\rho(z)| \geq \epsilon^{1/3} \right\} \leq c_3 \epsilon^{(1-4a)/12} \mathbb{P} \left\{ \tau_\epsilon < \infty \right\}.$$

It follows from Proposition 5.4 below that

$$\mathbb{P} \left\{ \lambda < \tau_\epsilon < \infty \mid |\tilde{h}_\rho(z)| \leq \epsilon^{1/3} \right\} = o(\epsilon^{2-d}),$$

establishing (42).

As for showing (43), from (41) we see that we need to show that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^* \left[e^{-\beta \tau_\epsilon} \Lambda_{\tau_\epsilon}^{1-4a} 1_{V_\epsilon} \right] = c^* \mathbb{E}^*[e^{-\beta T}].$$

On the event V_ϵ , we have $\tau_\epsilon - \tau_\rho = o(1)$. Hence, it suffices to show that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^* \left[e^{-\beta \tau_\rho} \Lambda_{\tau_\epsilon}^{1-4a} 1_{V_\epsilon} \right] = c^* \mathbb{E}^*[e^{-\beta T}].$$

The expectation on the left equals

$$\mathbb{E}^* \left[e^{-\beta \tau_\rho} \mathbb{E}^*[\Lambda_{\tau_\epsilon}^{1-4a} 1_{V_\epsilon} \mid \gamma_\rho] \right].$$

We now use the fact that as $\epsilon \rightarrow 0$, we get better and better approximation to the chordal case. In particular, since $\epsilon \ll \epsilon^{1/2}$ and $\mathbb{P}(V_\epsilon) \rightarrow 1$,

$$\mathbb{E}^* [\Lambda_{\tau_\epsilon}^{1-4a} 1_{V_\epsilon} \mid \gamma_\rho] \sim \mathbb{E}^*[S_\tau^{1-4a}],$$

where the right side is the $\mathbb{E}^*[S_\tau^{1-4a}]$ in the calculation (41). This gives

$$\mathbb{E}^* [\Lambda_{\tau_\epsilon}(z)^{1-4a} 1_{V_\epsilon} \mid \gamma_\rho] = c^* [1 + o(1)],$$

and hence,

$$\mathbb{E}^* \left[e^{-\beta \tau_\rho} \Lambda_{\tau_\epsilon}^{1-4a} 1_{V_\epsilon} \right] = c^* \mathbb{E}^* \left[e^{-\beta \tau_\rho} 1_{V_\epsilon} \right] [1 + o(1)].$$

Finally, the dominated convergence theorem implies that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^* \left[e^{-\beta \tau_\rho} 1_{V_\epsilon} \right] = \mathbb{E}^* \left[e^{-\beta T} \right] = \Phi(z).$$

5.1. Some lemmas about radial SLE. Here we discuss the results needed to prove (42). We state the following lemma proved in [7] which is a radial SLE analogue of a chordal SLE estimate proved in [1].

Lemma 5.1. *There exist $0 < c_1 < c_2 < \infty$ such that if $-\pi/2 \leq x \leq \pi/2$ and $r < |x|$, then if γ is radial SLE in \mathbb{H}^* ,*

$$c_1 (r/|x|)^{4a-1} \leq \mathbb{P} \{ \text{dist}(x, \gamma) \leq r \} \leq c_2 (r/|x|)^{4a-1}.$$

We give a corollary of this. Suppose $z = x + iy \in \mathbb{H}^*$ with $y \leq 1$. For every $\rho > 0$, there exists K such that if $\text{dist}(z, \gamma) \geq Ky$, then $\tilde{\Upsilon}_\infty(z) \geq (1 - \rho) \tilde{\Upsilon}_0(z)$. This gives the following. Recall that $S(x + iy) \asymp \Lambda(x + iy)$ for $y \leq 1$.

Lemma 5.2. *For every $\rho > 0$, there exists $c = c_\rho < \infty$ such that if $z = x + iy \in \mathbb{H}^*$ with $y > 0$, then*

$$\mathbb{P} \{ \tilde{\Upsilon}_\infty(z) \leq (1 - \rho) \tilde{\Upsilon}_0(z) \} \leq c S(z)^{4a-1},$$

where $S(z) = y/|z|$.

Proposition 5.3. *There exists $0 < c_1 < c_2 < \infty$ such that if $z = x + iy \in \mathbb{H}^*$ with $y \leq 1$ and $\epsilon \leq 1/2$, then*

$$c_1 S(z)^{4a-1} \epsilon^{2-d} \leq \mathbb{P} \{ \tilde{\Upsilon}_\infty(z) \leq \epsilon \tilde{\Upsilon}_0(z) \} \leq c_2 S(z)^{4a-1} \epsilon^{2-d}$$

where $S(z) = y/|z|$.

Proof. The lower bound for $|z| < 1/4$ can be established by comparison with chordal SLE_κ . For other z , we can reduce to the $|z| < 1/4$ case by noting that there exists $c > 0$ (independent of z) such that with probability at least c , $Z_t \leq 1/8$ and $S_t > cS(z)$ for some t . We will prove the harder upper bound.

Recall that $S(z) \asymp \Lambda(z)$, so we can use either on the right side (with appropriate constant). Without loss of generality we assume $\epsilon = 2^{-n}$ for positive integer n . The case $\epsilon = 1/2$ was proved in [7]. As before, let

$$\tau_\epsilon = \inf\{t : \tilde{\Upsilon}_t = \epsilon \tilde{\Upsilon}_0\}.$$

We will first find a δ such that the estimate holds for $|z| \leq \delta$. Consider the following sequence of stopping times:

$$\sigma_0 = \xi_0 = 0, \quad \sigma_1 = \inf\{t : |\gamma(t)| = 2^{-1}\}.$$

For $k \geq 1$, let

$$\eta^k(t) = \tilde{h}_{\sigma_k}(\gamma(t)), \quad \xi_k = \inf\left\{t \geq \sigma_k : |\eta^k(t) - \tilde{h}_{\sigma_k}(z)| \leq 2^{-m} \operatorname{Im}[\tilde{h}_{\sigma_k}(z)]\right\}.$$

Here $m \geq 2$ is a fixed integer which we will specify below. If $\xi_k = \infty$ we set $\xi_j = \sigma_j = \infty$ for $j > k$. If $\xi_k < \infty$, let

$$\gamma^k(t) = \tilde{h}_{\xi_k}(\gamma(t + \xi_k)), \quad z_k = \tilde{h}_{\xi_k}(z), \quad \sigma_{k+1} = \inf\left\{t \geq \xi_k : |\gamma^k(t)| = 2^{-1}\right\}.$$

The Koebe one-quarter theorem implies that

$$2^{-m-1} \tilde{\Upsilon}_{\sigma_k} \leq \tilde{\Upsilon}_{\xi_k} \leq \tilde{\Upsilon}_{\sigma_k}.$$

By considering the conformal image under \tilde{h}_{ξ_k} of the line segment of length $2^{-m} \operatorname{Im}(\tilde{h}_{\sigma_k}(z))$ between $\tilde{h}_{\sigma_k}(z)$ and $\eta^k(\xi_k)$ and using the Beurling estimate, we see that we can choose m (uniformly over k and z) so that $|z_k| \leq \operatorname{Im}(z) \leq |z|$. We fix such an m . Let E_k , $k = 0, 1, 2, \dots$, denote the event

$$E_k = \{\xi_k \leq \tau_\epsilon < \sigma_{k+1}\}.$$

We claim that there exists $c_0 < \infty$, $\delta > 0$, such that if $|z| \leq \delta$, then

$$(44) \quad \mathbb{P}[E_k] \leq c_0 2^k S(z)^{4a-1} |z|^{k\beta} \epsilon^{2-d}, \quad \beta = 2a - \frac{1}{2} > 0.$$

The $k = 0$ case (for any $\delta \leq 1$) follows from the chordal estimate and the absolute continuity of radial and chordal SLE $_\kappa$ up to time σ_1 . To be specific, let $c_0 < \infty$ be a constant such that for all $|z| \leq 1/8$,

$$(45) \quad \mathbb{P}\{\tau_\epsilon < \sigma_1\} \leq c_0 S(z)^{4a-1} \epsilon^{2-d}.$$

Let us consider the $k = 1$ case. Suppose that $z = x + iy$, $r \leq 1$, and $ry < \tilde{\Upsilon}_{\sigma_1} \leq 2ry$. If we consider $\gamma := \gamma[0, \sigma_1]$ as a chordal path in \mathbb{H} , then we claim that

$$(46) \quad S_{\sigma_1}(z) \leq c_1 r^{1/2} y.$$

To see this we consider $D = \mathbb{H} \setminus \gamma$ and let $g : D \rightarrow \mathbb{H}$ be a conformal transformation with $g(\gamma(\sigma_1)) = 0$, $g(\infty) = \infty$. Since the angle is a conformal invariant, S is comparable to the minimum of the probabilities that a Brownian motion starting at $g(z)$ exits \mathbb{H} at the positive or negative real axis, respectively. By conformal invariance, this is bounded above by the probability that a Brownian motion starting at z reaches the circle of radius $1/2$ without leaving D . By the Beurling estimate, the probability that it gets to the circle of radius $2y$ without leaving D is $O(r^{1/2})$ and given this the probability to reach the disk of radius $1/2$ is $O(y)$, provided that $\delta \leq 1/8$. This establishes (46).

Lemma 5.1 implies that

$$\mathbb{P}\{\xi_1 < \infty \mid 2^{-j}y < \tilde{\Upsilon}_{\sigma_1} \leq 2^{1-j}y\} \leq c S_{\sigma_1}(z)^{4a-1} \leq c_2 2^{-\beta j} y^{4a-1},$$

where $\beta = 2a - \frac{1}{2} > 0$. The $k = 0$ estimate implies that

$$\mathbb{P}\{\tilde{\Upsilon}_{\sigma_1} \leq 2^{1-j}y\} \leq c 2^{j(d-2)}.$$

Hence, we see that there exists c_2 such that

$$\begin{aligned} \mathbb{P}\{\xi_1 < \infty, 2^{-j}y < \tilde{\Upsilon}_{\sigma_1} \leq 2^{1-j}y\} &\leq c_2 2^{j(d-2)} 2^{-\beta j} y^{4a-1} \\ &= c_2 2^{j(d-2)} 2^{-\beta j} |z|^{4a-1} S(z)^{4a-1}. \end{aligned}$$

We now choose $\delta < 1/8$ sufficiently small so that

$$c_2 \delta^\beta 2^{m(2-d)} \sum_{j=1}^{\infty} 2^{-\beta j} \leq \frac{c_0}{2}.$$

Using (45), we see that

$$(47) \quad \mathbb{P}\left[E_1 \mid \xi_1 < \infty, 2^{-j}y < \tilde{\Upsilon}_{\sigma_1} \leq 2^{1-j}y\right] \leq c_0 2^{(m+j-n)(2-d)}.$$

By combining this with the last estimate and summing over j , we see that for $|z| \leq \delta$

$$\mathbb{P}[E_1] = \sum_{j=1}^{\infty} \mathbb{P}\left[E_1 \cap \{2^{-j}y < \tilde{\Upsilon}_{\sigma_1} \leq 2^{1-j}y\}\right] \leq 2^{-1} c_0 |z|^{2a-\frac{1}{2}} S(z)^{4a-1} \epsilon^{2-d}.$$

The case $k > 1$ is done inductively in the same way. The values m, δ, c_0 are fixed. If (44) holds for $k-1$, then (47) becomes

$$\mathbb{P}[E_k \mid \xi_1 < \infty, 2^{-j}y < \tilde{\Upsilon}_{\sigma_1} \leq 2^{1-j}y] \leq 2^{1-k} c_0 2^{(m+j-n)(2-d)}.$$

This establishes (44) and hence the proof is complete for $|z| \leq \delta$.

Now for z with $\text{Im}(z) \leq 1$ but $|z| > \delta$ let

$$\rho = \rho_r = \inf\{t : |z - \gamma(t)| \leq r\}.$$

By considering the line segment from z to $\gamma(\rho)$ and using conformal invariance and the Beurling estimate, we see there exists $\rho = \rho_\delta$ such that $|\tilde{h}_\rho(z)| \leq \delta$ for all $y \leq 1$. We fix such an $r < 1/10$ and first note that Lemma 5.2 implies that

$$\mathbb{P}\{\rho < \infty\} \leq c S(z)^{4a-1},$$

and the estimate above gives

$$\mathbb{P}\{\tau < \infty \mid \rho < \infty\} \leq c \epsilon^{2-d}$$

which finishes the proof. \square

Proposition 5.4. *For every $\rho > 0$, there exists $c < \infty$ such that if $\epsilon > 0$ and*

$$\lambda = \lambda_r = \inf\{t : |\gamma(t)| = r\},$$

then for $|z| \leq r$,

$$\mathbb{P}\{\tau_\epsilon < \infty, \tilde{\Upsilon}_\infty \leq (1-\rho) \tilde{\Upsilon}_\lambda\} \leq c S(z)^{4a-1} (|z|/r)^{4a-1} \epsilon^{2-d}.$$

Proof. We fix ρ and allow constants to depend on ρ . Without loss of generality, we assume that $\epsilon = 2^{-n}$. Let E_k denote the event

$$E_k = \{2^{-k} \tilde{\Upsilon}_0 < \tilde{\Upsilon}_\lambda \leq 2^{-k+1} \tilde{\Upsilon}_0\}.$$

As shown above in the proof of Proposition 5.3, there exists c such that on the event E_k ,

$$S_\lambda(z) \leq c 2^{-k/2} \text{Im}(z) r^{-1} \leq c 2^{-k/2} (|z|/r),$$

and if $k > 1$,

$$\mathbb{P}[E_k] \leq c S(z)^{4a-1} 2^{-k(2-d)}.$$

There exists c_1 such that

$$\mathbb{P}\{\tau_\epsilon < \infty, \tilde{\Upsilon}_\infty \leq (1-\rho) \tilde{\Upsilon}_\lambda \mid E_k\} \leq c_1 S_\lambda(z)^{4a-1} 2^{-(n-k)(2-d)}.$$

By combining these estimates and summing over k , we see that there exists c_2 such that

$$\mathbb{P}\{\tau_\epsilon < \infty, \tilde{\Upsilon}_\infty \leq (1 - \rho) \tilde{\Upsilon}_\lambda\} \leq c_2 (|z|/r)^{4a-1} S(z)^{4a-1} \epsilon^{2-d}$$

and the proof is complete. \square

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REFERENCES

- [1] T. Alberts and M. J. Kozdron. Intersection probabilities for a chordal SLE path and a semicircle *Electron. Comm. Probab.*, 13:448–460, 2008.
- [2] V. Beffara. The dimension of the SLE curves. *Ann. Probab.* 36:1421–1452, 2008.
- [3] N.-G. Kang and N. Makarov. Radial SLE martingale-observables. Preprint, 2012.
- [4] G. F. Lawler. *Conformally Invariant Processes in the Plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.
- [5] G. F. Lawler. Multifractal analysis of the reverse flow for the Schramm-Loewner evolution. In C. Bandt, P. Mörters, and M. Zähle, editors, *Fractal Geometry and Stochastics IV*, volume 61 of *Progress in Probability*, pages 73–107. Birkhäuser, Basel, Switzerland, 2009.
- [6] G. F. Lawler. Schramm-Loewner Evolution (SLE). In S. Sheffield and T. Spencer, editors, *Statistical Mechanics*, volume 16 of *IAS/Park City Mathematics Series*, pages 231–296. American Mathematical Society, Providence, RI, 2009.
- [7] G. F. Lawler. Continuity of radial SLE and two-sided radial SLE at the terminal point. Preprint, 2011. Available online at [arXiv:1104.1620](https://arxiv.org/abs/1104.1620).
- [8] G. F. Lawler and B. M. Werner. Multi-point Green’s functions for SLE and an estimate of Beffara. To appear, *Ann. Probab.* Available online at [arXiv:1011.3551](https://arxiv.org/abs/1011.3551).
- [9] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. Math.*, 161:883–924, 2005.
- [10] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

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